3.3 Combining Gaussian variables

Suppose X is a datum drawn from a Gaussian distribution of population standard deviation σ_x, and similarly for Y. The distribution of $Z = X + Y$ follows very easily from the convolution of the two Gaussians: - especially if we remember the Convolution Theorem, which tells us that the Fourier transform of a convolution is just the product of the transforms of the parts of the convolution (with appropriate complex conjugates). In probability and statistics, Fourier transforms are called "characteristic functions" sometimes.

The Fourier transform of the Gaussian is

$$\int dx\, e^{\imath kx} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and is just

$$e^{\imath \mu k} e^{-\frac{\sigma^2 k^2}{2}}$$

so it is another Gaussian, of width equal to the reciprocal of the width of the original. This is a feature of Fourier transforms.

Because of our definition of the transform, it is equal to unity at $k = 0$, reflecting the normalization of a probability distribution.

Let's call the two distributions in our problem g_x and g_y - the convolution result says

$$g_z(z) = \int dt\, g_x(t-z) g_y(t)$$

so, using upper case for the transforms, the convolution theorem says

$$G_z = G_x G_y^*$$

or

$$G_z = e^{-\frac{(\sigma_x^2 + \sigma_y^2)k^2}{2}}$$

again, correctly normalized. We can invert this transform again easily to get

$$g_z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_y^2)}} e^{-\frac{z^2}{2(\sigma_x^2 + \sigma_y^2)}}$$

This tells us that, exactly for the Gaussian, we can add errors in quadrature. For other distributions this familiar result is approximate only.